Schur-Weyl Duality, Fused Algebras and Affine Hecke Algebras

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Plan

Historical Context and Motivation

- Representation theory.
- Late 19th century.
- Schur-Weyl duality (early 20th century).
- Generalization through a recent theory (late 20th century to present) : the fused Hecke algebras.
- Constructing irreducible representations of $GL(V)$ using those of the symmetric group.
- Decomposing $V^{\otimes k}$, $k \geq 3$.

Definitions

Let $\mathscr{A}$ an (unital) algebra over $\mathbb{C}$.

Definition

A **representation** of $\mathscr{A}$ is the data of (V, ρ) such that :

- V is a $\mathbb{C}$-vector space.
- $\rho : \mathscr{A} \longrightarrow End_{\mathbb{C}}(V)$ is an algebra morphism, *i.e.* a linear application such that :

$$\forall a, b \in \mathscr{A}, \quad \rho(a.b) = \rho(a) \circ \rho(b),$$

$$\rho(1_{\mathscr{A}}) = Id_V.$$

Examples :

- $\rho : \mathscr{A} \longrightarrow End_{\mathbb{C}}(V), \quad a \longmapsto Id_V$ $\rightsquigarrow$ trivial representation.
- $\rho : \mathscr{A} \longrightarrow End_{\mathbb{C}}(\mathscr{A}), \quad a \longmapsto (b \longmapsto a.b)$ $\rightsquigarrow$ regular representation.

Definition

A representation (ρ, V) of $\mathscr{A}$ is called **irreducible** if :

$$\{0_V\} \neq W \subset V \text{ s.t. } \rho(\mathscr{A})(W) \subseteq W \Longrightarrow W = V.$$

Example : A one-dimensional representation.

Symmetric group algebra

Definition

The **symmetric group algebra** $\mathbb{C}\mathfrak{S}_n$ is the algebra with :

- Basis indexed by $\mathfrak{S}_n$, *i.e.* $\{e_g\}_{g\in\mathfrak{S}_n}$.
- Multiplication is the group multiplication, *i.e.* :

$$\forall g,h\in\mathfrak{S}_n, \quad e_g e_f = e_{gf}.$$

Example : $n=3$: $\mathbb{C}\mathfrak{S}_3 = Vect(\{1, e_{(1\ 2)}, e_{(2\ 3)}, e_{(1\ 3)}, e_{(1\ 2\ 3)},\ e_{(1\ 3\ 2)}\})$.
$\dim(\mathbb{C}\mathfrak{S}_3)=6$.
$e_{(1\ 2\ 3)}e_{(2\ 3)} = e_{(1\ 2)}$.
Remarks :

- $\dim(\mathbb{C}\mathfrak{S}_n) = |\mathfrak{S}_n| = n!$.
-
$$\{\text{Representations of } \mathfrak{S}_n\} \underset{1:1}{\leftrightarrow} \{\text{Representations of } \mathbb{C}\mathfrak{S}_n\}.$$
$$\{\text{Irreducibles of } \mathfrak{S}_n\} \underset{1:1}{\leftrightarrow} \{\text{Irreducibles of } \mathbb{C}\mathfrak{S}_n\}.$$
- A representation of $\mathbb{C}\mathfrak{S}_n$ is a direct sum of irreducible ones.$\rightsquigarrow$ $\mathbb{C}\mathfrak{S}_n$ is **semisimple**.
$\rightsquigarrow$ understanding of $Irr(\mathbb{C}\mathfrak{S}_n)$=understanding of $\mathbb{C}\mathfrak{S}_n$.

Let $n \in \mathbb{Z}_{\geq 1}$.

Definition

A **partition** of n is a sequence $\lambda := (\lambda_1, \ldots, \lambda_\ell)$ such that $\lambda_1 \geq \cdots \geq \lambda_\ell \geq 1$ and $\lambda_1 + \cdots + \lambda_\ell = n$.

Examples :

- $n = 3$: (3), $(2,1)$ and $(1,1,1)$.
- $n = 4$: (4), $(3,1)$, $(2,2)$, $(2,1,1)$ and $(1,1,1,1)$.

Theorem

A complete set of pairwise non-isomorphic irreducible representations of $\mathbb{C}\mathfrak{S}_n$ is indexed by the partitions of n, *i.e.* :

$$Irr(\mathbb{C}\mathfrak{S}_n) = \{V_\lambda \mid \lambda \vdash n\}.$$

Examples :

- $n = 3$: 3 irreducible representations.
- $n = 4$: 5 irreducible representations.

Plan

Commutant of a subset

Let $P \in M_n(\mathbb{C})$.

$$Comm(P) = \{M \in M_n(\mathbb{C}) \mid [P, M] = PM - MP = 0\}.$$

Example :

$$P = \left(\begin{array}{c|c|c} (\lambda_1) & 0 & 0 \\ \hline 0 & \ddots & 0 \\ \hline 0 & 0 & (\lambda_p) \end{array}\right) \rightsquigarrow M \in Comm(P) \Leftrightarrow M = \left(\begin{array}{c|c|c} A_1 & 0 & 0 \\ \hline 0 & \ddots & 0 \\ \hline 0 & 0 & A_p \end{array}\right).$$

$\rightsquigarrow$ The commutant $Comm(P)$ "knows" about the multiplicities.

Commutant of a subset

Let $S = \{N_1, \dots, N_\ell\}$.

$$Comm(S) = \{M \in M_n(\mathbb{C}) \mid \forall i = 1, \dots, \ell,\ [N_i, M] = 0\}.$$

Example : S a commutative set of diagonalizable matrices.

$$S \text{ is a complete set} \Leftrightarrow Comm(S) \text{ is commutative.}$$

$\rightsquigarrow$ commutant "knows" about completeness. $\rightsquigarrow$ provides a way to complete the set.

Commutant of a subset

(V,ρ) representation of $\mathscr{A} \rightsquigarrow \rho : \mathscr{A} \longrightarrow M_n(\mathbb{C})$.

$$Comm(\rho(\mathscr{A})) := \{M \in M_n(\mathbb{C}) \mid \forall a \in \mathscr{A},\ [\rho(a), M] = 0\}.$$

$$V \simeq V_1^{\oplus m_1} \oplus \cdots \oplus V_p^{\oplus m_p}.$$

Lemma (Schur's lemma)

$$(V,\rho) \text{ is irreducible } \Leftrightarrow Comm(\rho(\mathscr{A})) = \{\lambda Id_V,\ \lambda \in \mathbb{C}\}.$$

Corollary

- One-multiplicity :

$$m_i = 1 \text{ for all } i = 1,\dots,p \Leftrightarrow Comm(\rho(\mathscr{A})) \text{ is commutative.}$$

- More generally :

$$Comm(\rho(\mathscr{A})) \cong M_{m_1}(\mathbb{C}) \oplus \cdots \oplus M_{m_p}(\mathbb{C}).$$

$\rightsquigarrow$ $Comm(\rho(\mathscr{A}))$ "knows" the multiplicities.

Commutant of a subset

$$\begin{aligned} V &\simeq V_1^{\oplus m_1} \quad \oplus \quad \dots \quad \oplus \quad V_p^{\oplus m_p} \\ &\simeq V_1 \otimes M_1 \quad \oplus \quad \dots \quad \oplus \quad V_p \otimes M_p. \end{aligned}$$

From the point of view of $\mathscr{A}$:

$$V \simeq V_1^{\oplus \dim(M_1)} \quad \oplus \quad \dots \quad \oplus \quad V_p^{\oplus \dim(M_p)}.$$

From the point of view of $Comm(\rho(\mathscr{A}))$:

$$V \simeq M_1^{\oplus \dim(V_1)} \quad \oplus \quad \dots \quad \oplus \quad V_p^{\oplus \dim(V_p)}.$$

Multiplicities for $\mathscr{A}$ = Dimensions for $Comm(\rho(\mathscr{A}))$.

Multiplicities for $Comm(\rho(\mathscr{A}))$ = Dimensions for $\mathscr{A}$.

Schur-Weyl duality

Definition

$$\begin{array}{rccc} \rho_k : & GL(V) & \longrightarrow & End(V^{\otimes k}) \\ & g & \longmapsto & \left\{ \begin{array}{ccc} V^{\otimes k} & \longrightarrow & V^{\otimes k} \\ v_1 \otimes \ldots \otimes v_k & \longmapsto & gv_1 \otimes \ldots \otimes gv_k \end{array} \right. \end{array}$$

$$\begin{array}{rccc} \sigma_k : & \mathfrak{S}_k & \longrightarrow & End(V^{\otimes k}) \\ & s & \longmapsto & \left\{ \begin{array}{ccc} V^{\otimes k} & \longrightarrow & V^{\otimes k} \\ v_1 \otimes \ldots \otimes v_k & \longmapsto & v_{s^{-1}(1)} \otimes \ldots \otimes v_{s^{-1}(k)} \end{array} \right. \end{array}$$

Definition

$$Comm(S) = \{x \in End(V) \mid x \circ s = s \circ x \quad \forall s \in S\}.$$

$\mathscr{A} \subset End(V^{\otimes k})$ a semisimple algebra, $\mathscr{B} := Comm(\mathscr{A})$.

$$\bigoplus_{i=1}^{k} V_i^{\oplus \dim(U_i)} \quad \cong \quad V^{\otimes k} \quad \cong \quad \bigoplus_{i=1}^{k} U_i^{\oplus \dim(V_i)},$$

$$V^{\otimes k} \quad \cong \quad \bigoplus_{i=1}^{k} V_i \otimes U_i.$$

Schur-Weyl duality

$$\mathbb{C}\mathfrak{S}_k \xrightarrow{\sigma_k} End\left(V^{\otimes k}\right) \xleftarrow{\rho_k} \mathbb{C}GL(V)$$

Let $\mathscr{A} := \sigma_k(\mathbb{C}\mathfrak{S}_k)$, $\mathscr{B} := \rho_k(\mathbb{C}GL(V))$.

Theorem (Schur)

$\mathscr{A} = Comm(\mathscr{B})$ and $\mathscr{B} = Comm(\mathscr{A})$.

Theorem (Schur-Weyl duality)

$$\bigotimes^k V = \bigoplus_{\substack{\lambda \in \mathscr{P}_k \\ \ell(\lambda) \leq \dim(V)}} V_\lambda \otimes \mathbb{S}_\lambda(V).$$

Extension of the duality

Let $n \in \mathbb{Z}_{\geq 0}$, $\underline{k} = (k_1, \dots, k_n)$ and $N := \sum_{i=1}^{n} k_i$. Let $P_{\underline{k},n}$ (idempotent in $\mathbb{C}\mathfrak{S}_N$).

Definition

For $n \in \mathbb{Z}_{\geq 1}$ and $\underline{k} = (k_1, k_2, \dots, k_n) \in (\mathbb{Z}_{\geq 1})^n$, we define the algebra :

$$\mathscr{H}_{\underline{k},n}(1) := P_{\underline{k},n} \cdot \mathbb{C}\mathfrak{S}_N \cdot P_{\underline{k},n}.$$

$$\mathscr{H}_{\underline{k},n}(1) \xrightarrow{\sigma_{\underline{k},n}} End\left(\bigotimes_{i=1}^{n} Sym^{k_i}(V)\right) \xleftarrow{\rho_{\underline{k},n}} \mathbb{C}GL(V)$$

Set $\mathscr{A} := \sigma_{\underline{k},n}\left(\mathscr{H}_{\underline{k},n}(1)\right)$ and $\mathscr{B} := \rho_{\underline{k},n}(\mathbb{C}GL(V))$.

Theorem

$Comm(\mathscr{A}) = \mathscr{B}$ and $Comm(\mathscr{B}) = \mathscr{A}$.

Theorem (Extension of Schur-Weyl duality)

$$\bigotimes_{i=1}^{n} Sym^{k_i}(V) \cong \bigoplus_{\substack{\lambda \in U_{\underline{k},n} \\ \ell(\lambda) \leq \dim(V)}} \underbrace{\mathbb{S}_\lambda(V) \otimes W_\lambda.}_{\mathbb{C}GL(V) \otimes H_{\underline{k},n}(1)-\text{module}}$$

Plan

Theorem (Artin-Wedderburn, 1907)

Finite-dimensional semi-simple algebras are finite direct sums of algebras of the form $End(V)$, where V is a finite-dimensional vector space. Hence, we have an algebra isomorphism :

$$\Phi : \mathscr{A} \xrightarrow{\cong} \bigoplus_{\lambda \in \Lambda} End(V^\lambda)$$

where Λ is a finite set.

Example : Algebra of a finite group.

Corollary

Let A be a finite-dimensional semi-simple algebra. Then we have :

$$\dim(\mathscr{A}) = \sum_{\lambda \in \Lambda} \dim(V^\lambda)^2.$$

Corollary (Dimension of the center)

$$\dim(Z(\mathscr{A})) = |\Lambda|.$$

Examples

The symmetric group algebra :

- $n=3$: $\mathbb{C}\mathfrak{S}_3 \cong \mathbb{C}^{\oplus 2} \oplus M_2(\mathbb{C})$.
- $n=4$: $\mathbb{C}\mathfrak{S}_4 \cong \mathbb{C}^{\oplus 2} \oplus M_2(\mathbb{C}) \oplus M_3(\mathbb{C})^{\oplus 2}$.
- $n=5$: $\mathbb{C}\mathfrak{S}_5 \cong \mathbb{C}^{\oplus 2} \oplus M_4(\mathbb{C})^{\oplus 2} \oplus M_5(\mathbb{C})^{\oplus 2} \oplus M_6(\mathbb{C})$.

The algebra $\mathscr{H}_{\underline{k},n}(1)$:

Theorem

The algebra $\mathscr{H}_{\underline{k},n}(1) := P_{\underline{k},n}\mathbb{C}\mathfrak{S}_N P_{\underline{k},n}$ is semisimple.

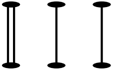
- $n=2$, any $\underline{k} \in (\mathbb{N}^*)^2$: $\mathscr{H}_{\underline{k},2}(1) \cong \mathbb{C}^{\oplus N}, \quad N \in \mathbb{N}^*$.
- $n=3$, $\underline{k} = (2,2,2)$: $\mathscr{H}_{\underline{k},3}(1) \cong \mathbb{C}^{\oplus 4} \bigoplus M_2(\mathbb{C})^{\oplus 2} \bigoplus M_3(\mathbb{C})$.
- $n=4$, $\underline{k} = (2,2,2,2)$: $\mathscr{H}_{\underline{k},4}(1) \cong \mathbb{C}^{\oplus 3} \bigoplus M_2(\mathbb{C}) \bigoplus M_3(\mathbb{C})^{\oplus 6} \bigoplus M_6(\mathbb{C})^{\oplus 3} \bigoplus M_7(\mathbb{C}) \bigoplus M_8(\mathbb{C})$.

Plan

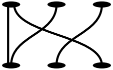
Fused Permutations Algebra

Examples :

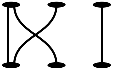
- $n = 3$, $\underline{k} = (2,1,1)$. 6 possibilities :



$(\{1,1\},\{2\},\{3\})$

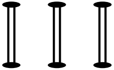


$(\{1,3\},\{1\},\{2\})$

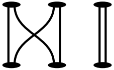


$(\{1,2\},\{2\},\{3\})$.

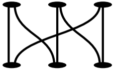
- $n \in \mathbb{N}^*$, $\underline{k} = (1,1,\ldots,1)$. Fused permutation $\leftrightsquigarrow$ Permutation.
- $n = 3$, $\underline{k} = (2,2,2)$. 21 diagrams :



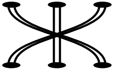
$(\{1,1\},\{2,2\},\{3,3\})$



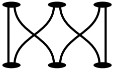
$(\{1,2\},\{1,2\},\{3,3\})$



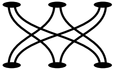
$(\{1,2\},\{2,3\},\{1,3\})$



$(\{3,3\},\{2,2\},\{1,1\})$



$(\{1,2\},\{1,3\},\{2,3\})$



$(\{2,3\},\{1,3\},\{1,2\})$.

Fused Permutations Algebra

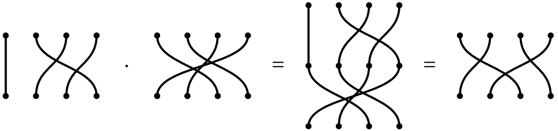
Définition

Fused Permutations Algebra (FPA) $(H_{\underline{k},n}(1),+,\cdot)$:

- Vector space : Basis indexed by the fused permutations.
- Multiplication : *Cf.* Example.
  Let d,d' 2 fused permutations.
  - (Concatenation) We place the diagram of d on top of the diagram of d' by identifying the bottom ellipses of d with the top ellipses of d'.
  - (Removal of middle ellipses) For each $a\in\{1,\dots,n\}$, there are k_a edges arriving and k_a edges leaving the a-th ellipse in the middle row. So for each $a\in\{1,\dots,n\}$, we delete the a-th ellipse in the middle row and sum over all possibilities of connecting the k_a edges arriving at it to the k_a edges leaving from it (at the a-th edge, there are thus $k_a!$ possibilities).
  - (Normalisation) We divide the resulting sum by $k_1!\dots k_n!$.

Example : $n=4$, $\underline{k}=(1,1,1,1)$:

$(2\ 3\ 4)\circ(1\ 4\ 2\ 3)=$

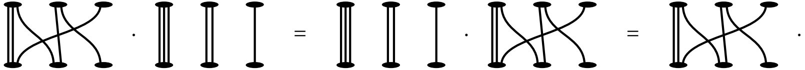


$=(1\ 2\ 4\ 3).$

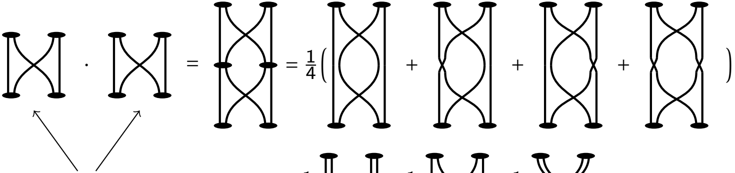
Fused Permutations Algebra

Examples :

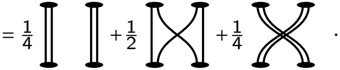
- $n = 3,\ \underline{k} = (3,2,1)$:



- $n = 2,\ \underline{k} = (2,2)$:



$(\{1,2\},\{1,2\})$



Theorem

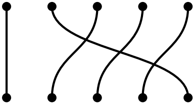
The algebras $\mathscr{H}_{\underline{k},n}(1) := P_{\underline{k},n}\mathbb{C}\mathfrak{S}_N P_{\underline{k},n}$ and $H_{\underline{k},n}(1)$ are isomorphic as algebras.

$$\mathrm{H}: \begin{array}{ccc} \mathscr{H}_{\underline{k},n}(1) & \longrightarrow & H_{\underline{k},n}(1) \\ P_{\underline{k},n}\omega P_{\underline{k},n} & \longmapsto & [\omega] \end{array} \rightsquigarrow \text{isomorphism of algebras.}$$

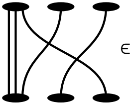
Example : $n=3$ and $\underline{k}=(3,1,1)$. Let $\omega = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 4 & 5 & 2 \end{pmatrix}$

$\omega(i)$

i



$\in \mathfrak{S}_{3+1+1}$.

$\mathrm{H}(P_{\underline{k},n}\omega P_{\underline{k},n}) = [\omega] =$  $\in H_{(2,1,1),3}$.

Chain of algebras as n varies :

$$\mathbb{C} = H_{\underline{k},0}(1) \subset H_{\underline{k},1}(1) \subset H_{\underline{k},2}(1) \subset \cdots \subset H_{\underline{k},n}(1) \subset H_{\underline{k},n+1}(1) \subset \cdots.$$

Questions

:

- Generators of $H_{\underline{k},n}$?
- Canonical basis for the chain $(H_{\underline{k},n})_{n\geq 0}$? For the center ?
- Presentation by generators and relations ?

TOO HARD IN GENERAL !

$\rightsquigarrow$ particular cases for the sequences $\underline{k}$.

Plan

CDAHA

Let $n \in \mathbb{Z}_{\geq 0}$. Fix a pair $\kappa = (\kappa_1, \kappa_2) \in \mathbb{C}^2$.

Definition

The **cyclotomic degenerate affine Hecke algebra** (CDAHA) with cyclotomic parameter κ is the unital associative algebra $\widehat{\mathscr{H}}_n^\kappa$ generated by $x_1, x_2, \ldots, x_n$ (Jucys-Murphy elements), $s_1, \ldots, s_{n-1}$ (Coxeter elements), subject to the following relations :

$$(x_1 - \kappa_1)(x_1 - \kappa_2) = 0,$$

$$x_1 s_\ell = s_\ell x_1 \quad \text{for} \quad \ell = 2, \ldots, n-1,$$

$$x_u x_t = x_t x_u, \quad u, t = 1, \ldots, n,$$

$$x_{r+1} = s_r x_r s_r + s_r, \quad r = 1, \ldots, n-1,$$

$$s_k^2 = 1 \quad \text{for} \quad k = 1, \ldots, n-1,$$

$$s_i s_j = s_j s_i \quad \text{if} \quad |i-j| > 1,$$

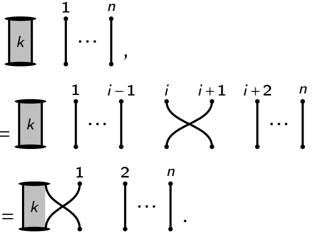
$$s_k s_{k+1} s_k = s_{k+1} s_k s_{k+1} \quad \text{for} \quad k = 1, \ldots, n-2.$$

Theorem (H, M, 2014)

Let $\kappa = (0, k+1)$. The algebra $\widehat{\mathscr{H}}_n^\kappa$ is semisimple if and only if $n \leq k+1$.

Let $\underline{k} = (k, 1, \dots, 1)$, $k \in \mathbb{Z}_{\geq 1}$.
Examples of diagrams in $H_{\underline{k},n}(1)$:

$1 :=$, $\sigma_i :=$, $i = 1, \dots, n-1$, $\sigma_0 :=$.



Proposition (D., 2025)

$H_{\underline{k},n}(1)$ is a quotient of $\widehat{\mathscr{H}}_n^{\kappa}$ for $\kappa = (0, k+1)$.

$$
\begin{array}{rccc}
\varphi : & \widehat{\mathscr{H}}_n^{\kappa} & \longrightarrow & H_{\underline{k},n}(1) \\
 & x_1 & \longmapsto & t := 1 + k\sigma_0, \\
 & s_i & \longmapsto & \sigma_i, \qquad\qquad i = 1, \dots, n-1.
\end{array}
$$

Connection with the FPA

Let $n \leq k$. We set :

$$Par_2(n) := \{\boldsymbol{\lambda} = (\lambda_1, \lambda_2) \mid |\lambda_1| + |\lambda_2| = n\}.$$

Example : $n = 2$: $Par_2(2) = \{((2), \emptyset), ((1,1), \emptyset), ((1),(1)), (\emptyset, (1,1)), (\emptyset, (2))\}$.

Proposition

$$Irr(\widehat{\mathscr{H}}_n^{\mathbb{K}}) = \{V^{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \in Par_2(n)\}.$$

From Artin-Wedderburn theorem :

$$\widehat{\mathscr{H}}_n^{\mathbb{K}} \cong \bigoplus_{\boldsymbol{\lambda} \in Par_2(n)} End(V^{\boldsymbol{\lambda}}).$$

Corollary

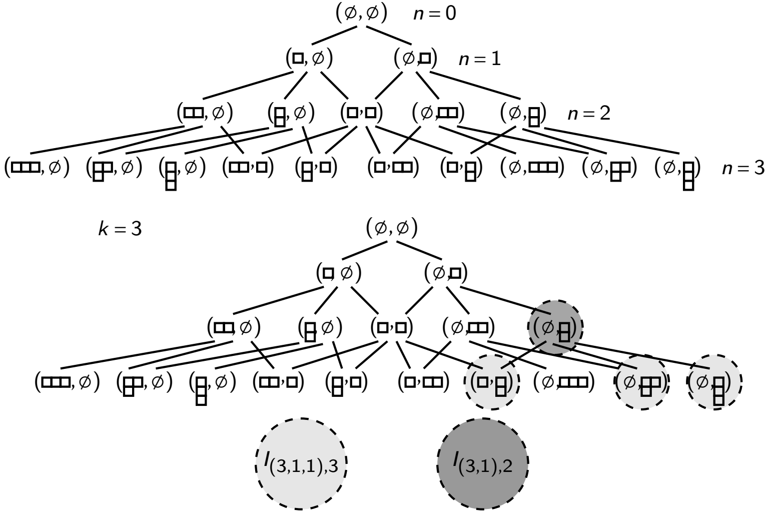
There exists a subset $S \subset Par_2(n)$ such that, for $n \leq k$, we have :

$$H_{\underline{k},n}(1) \cong \bigoplus_{\boldsymbol{\lambda} \in S} End(V^{\boldsymbol{\lambda}}).$$

Questions :

- Which sets have to be removed ? What is exactly the subset $S \subset Par_2(n)$?
- What is the kernel of the surjective map $\varphi : \widehat{\mathscr{H}}_n^{\mathbb{K}} \longrightarrow H_{\underline{k},n}(1)$?

Bratelli diagram



Result and Conjecture

Let $n \in \mathbb{Z}_{\geq 1}$ and $p \geq 0$. Let $\kappa = (0, n+p+1)$.

Theorem (D., 2025)

A presentation by generators and relations of $H_{\underline{n+p},n}(1)$ is given by :

- Generators : $x_1, \ldots, x_n, s_1, \ldots, s_{n-1}$.
- Relations :
  1. Defining relations of the CDAHA.
  2. $(x_1 s_1)^2 = x_1 s_1 x_1 - x_1 s_1 + x_1$.

In particular, $H_{\underline{n},n}(1) \cong H_{\underline{n+p},n}(1)$.

What about $k \leq n-1$?

Theorem (D., 2025)

A presentation by generators and relations of $H_{\underline{n-1},n}(1)$ is given by :

- Generators : $x_1, \ldots, x_n, s_1, \ldots, s_{n-1}$.
- Relations :
  1. Defining relations of the CDAHA.
  2. $(x_1 s_1)^2 = x_1 s_1 x_1 - x_1 s_1 + x_1$.
  3. $E_{(\square \ldots \square\ ,\ \emptyset)} = 0$.

Canonical basis

Consider the following product of sets :

$$\left\{\begin{matrix}1,\\ x_1\end{matrix}\right\}\cdot\left\{\begin{matrix}1,\\ s_1,\\ s_1x_1,\\ s_1x_1s_1\end{matrix}\right\}\cdot\left\{\begin{matrix}1,\\ s_2,\\ s_2s_1,\\ s_2s_1x_1,\\ s_2s_1x_1s_1,\\ s_2s_1x_1s_1s_2\end{matrix}\right\}\cdot\ \dots\ \cdot\left\{\begin{matrix}1,\\ s_{n-1},\\ \vdots\\ s_{n-1}\dots s_1x_1,\\ s_{n-1}\dots s_1x_1s_1,\\ \vdots\\ s_{n-1}\dots s_1x_1s_1\dots s_{n-1}\end{matrix}\right\}.$$

$\rightsquigarrow$ basis set of $\widehat{\mathscr{H}}_n^{\kappa}$. In particular, $\dim(\widehat{\mathscr{H}}_n^{\kappa}) = 2^n n!$.

Definition

The **signed permutations group** B_n is the group of permutations σ of $\{-n, -(n-1), \dots, -1, 1, \dots, n-1, n\}$ such that $\sigma(-i) = -\sigma(i)$, $i = 1, \dots, n$.

$\sigma \rightsquigarrow b_1 \dots b_n$, $b_i \in \{1, \overline{1}, \dots, n, \overline{n}\}$.

Example : $n = 3$: $\overline{2}1\overline{3} \leftrightsquigarrow \begin{pmatrix}-3 & -2 & -1 & 1 & 2 & 3\\ 3 & -1 & 2 & -2 & 1 & -3\end{pmatrix}$.

Canonical basis

Generators of B_n : τ_i, where $\tau_0 = (-1\ 1)$ and $\tau_i = (i\ i+1)$, $i = 1, \ldots, n-1$.

$$B_n = \left\{ \begin{array}{c} 1, \\ \tau_0 \end{array} \right\} \cdot \left\{ \begin{array}{c} 1, \\ \tau_1, \\ \tau_1\tau_0, \\ \tau_1\tau_0\tau_1 \end{array} \right\} \cdot \left\{ \begin{array}{c} 1, \\ \tau_2, \\ \tau_2\tau_1, \\ \tau_2\tau_1\tau_0, \\ \tau_2\tau_1\tau_0\tau_1, \\ \tau_2\tau_1\tau_0\tau_1\tau_2 \end{array} \right\} \cdot \ \ldots \ \cdot \left\{ \begin{array}{c} 1, \\ \tau_{n-1}, \\ \vdots \\ \tau_{n-1}\ldots\tau_1\tau_0, \\ \tau_{n-1}\ldots\tau_1\tau_0\tau_1, \\ \vdots \\ \tau_{n-1}\ldots\tau_1\tau_0\tau_1\ldots\tau_{n-1} \end{array} \right\}.$$

$\rightsquigarrow \widehat{\mathscr{H}}_n^{\kappa}$ has a basis indexed by elements of B_n.
Consequence : A basis of $H_{\underline{k},n}$ is given by a subset of the images of basis elements of $\widehat{\mathscr{H}}_n^{\kappa}$ under the surjective morphism $\varphi : \widehat{\mathscr{H}}_n^{\kappa} \longmapsto H_{\underline{k},n}$.
Question : What is the subset of B_n that index the basis of $H_{\underline{k},n}$?

Definition

We define the subset $B_n(\overline{12}, \overline{k+1\ldots 1})$ of B_n as the set of words with no increasing sequence of barred numbers and at most k barred integers.

Example :

- $261\overline{3}4\overline{5} \notin B_3(\overline{12})$.
- $2\overline{54}1\overline{3}6 \in B_6(\overline{12}, \overline{4321})$ but is not in $B_6(\overline{12}, \overline{321})$.

We have the following identity :

$$|B_n(\overline{12}, \overline{k+1\ldots 1})| = \sum_{i=0}^{\min(k,n)} \binom{n}{i}^2 (n-i)!.$$

Remark : $|B_n(\overline{12}, \overline{k+1\ldots 1})| = \dim(H_{\underline{k},n})$.

Theorem (D. 2025)

$H_{\underline{k},n}$ has a basis indexed by the set $B_n(\overline{12}, \overline{k+1\ldots 1})$.

Plan

Short-term goals :

- Canonical basis for the chains of algebras $(H_{\underline{k},n}(1))_{n\geq 1}$ and $(Z(H_{\underline{k},n}(1)))_{n\geq 1}$, when $\underline{k}=(k,k,\dots,k)$.
- Study of Jucys-Murphy elements for $(H_{\underline{k},n}(1))_{n\geq 1}$.

Long-term goals :

- Generalize to an arbitrary $\underline{k}$.
- Understand the centralizer $End_{GL(V)}(\bigotimes_{i=1}^{n} Sym^{k_i}(V)) = \sigma_{\underline{k},n}(H_{\underline{k},n}(1))$ as a quotient of $H_{\underline{k},n}(1)$.
- Applications.

Thank you for your attention !

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