

The classical Schur-Weyl duality and one of its generalizations

$$\mathbb{C}\mathfrak{S}_k \xrightarrow{\sigma} \text{End}(V^{\otimes k}) \xleftarrow{\rho} \text{CGL}(V)$$

$$\text{Comm}(\sigma(\mathbb{C}\mathfrak{S}_k)) = \rho(\text{CGL}(V))$$

$$\text{Comm}(\rho(\text{CGL}(V))) = \sigma(\mathbb{C}\mathfrak{S}_k)$$

$$\mathbb{C}\mathfrak{S}_{|\underline{k}|} \longrightarrow P_{\underline{k},n} \mathbb{C}\mathfrak{S}_{\underline{k}} P_{\underline{k},n}$$

$$\cong H_{\underline{k},n}$$



$$H_{\underline{k},n} \xrightarrow{\sigma} \text{End}\left(\bigotimes_{i=1}^n S^{k_i} V\right) \xleftarrow{\rho} \text{CGL}(V)$$

$$\text{Comm}(\sigma(H_{\underline{k},n})) = \rho(\text{CGL}(V))$$

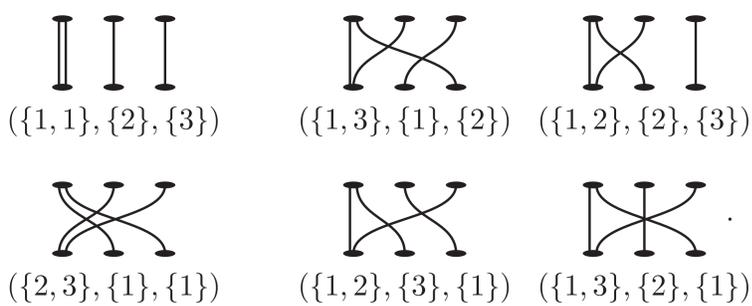
$$\text{Comm}(\rho(\text{CGL}(V))) = \sigma(H_{\underline{k},n})$$

The algebra $H_{\underline{k},n}$

The fused permutations algebra $H_{\underline{k},n}$ is defined as:

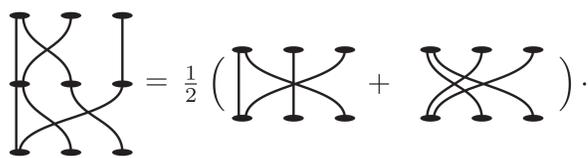
- Diagrams as basis elements: **the fused permutations**.

Example for $n = 3$ and $\underline{k} = (2, 1, 1)$:



- Multiplication: Generalization of the one of the symmetric group algebra by summing over all possibilities on the edges and then renormalization.

Example:



Representation theory

$$\text{Irr}(H_{\underline{k},n}) = \{W_{\underline{k},\lambda} \mid \lambda \vdash |\underline{k}|, \underline{k}^{\text{ord}} \leq \lambda\}.$$

Artin-Wedderburn theorem:

$$H_{\underline{k},n} \cong \bigoplus \text{End}(W_{\underline{k},\lambda}).$$

Example for $\underline{k} = (2, 1, 1)$:

$$H_{(2,1,1),3} \cong \text{End}(W_{(2,1,1),(4)}) \oplus \text{End}(W_{(2,1,1),(3,1)}) \oplus \text{End}(W_{(2,1,1),(2,2)}).$$

Link with the Kostka numbers

Basis of each irreducible representation $W_{\underline{k},\lambda}$ is indexed by the semistandard tableaux of shape λ and weight \underline{k} .

In particular, $\dim(W_{\underline{k},\lambda}) = K_{\lambda,\underline{k}}$.

The following identity holds:

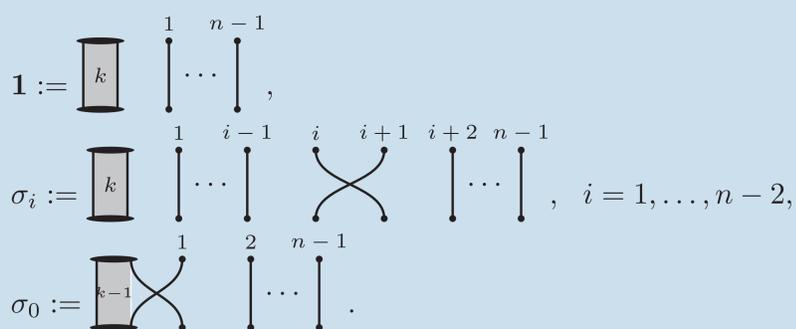
$$\dim(H_{\underline{k},n}) = \sum_{\lambda \vdash |\underline{k}|} K_{\lambda,\underline{k}}^2.$$

Example for $\underline{k} = (2, 1, 1)$: $\dim(W_{(2,1,1),(3,1)}) = K_{(3,1),(2,1,1)} = 2$.

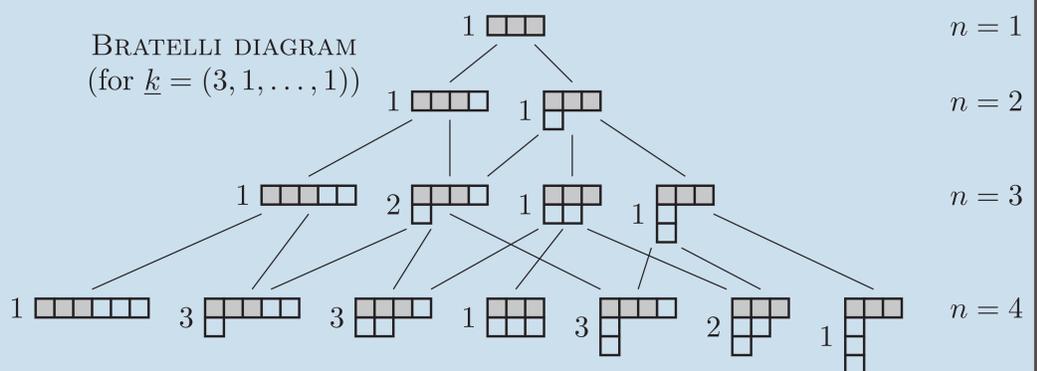
$$\text{SSTab}((3,1), (2,1,1)) = \left\{ \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 3 & & \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 1 & 1 & 3 \\ \hline 2 & & \\ \hline \end{array} \right\}.$$

One-boundary case $\underline{k} = (k, 1, 1, \dots, 1)$.

GENERATORS OF $H_{(k,1,\dots,1),n}$:



BRATELLI DIAGRAM
(for $\underline{k} = (3, 1, \dots, 1)$)



Connection with the affine Hecke algebra

There exists an algebra called the **cyclotomic degenerate affine Hecke algebra of level 2**, that is denoted $\hat{\mathcal{H}}_{n-1}^{(0,k+1)}$ (see [2]), and a surjective map from $\hat{\mathcal{H}}_{n-1}^{(0,k+1)}$ to $H_{(k,1,\dots,1),n}$:

$$\varphi: \hat{\mathcal{H}}_{n-1}^{(0,k+1)} \twoheadrightarrow H_{(k,1,\dots,1),n}.$$

Moreover, the kernel of φ is well-known (see [1]).

Consequence on the Schur-Weyl duality

The kernel of the surjective map

$$\sigma: H_{\underline{k},n} \twoheadrightarrow \text{End}_{\text{CGL}(V)}\left(\bigotimes_{i=1}^n \text{End}(S^{k_i} V)\right)$$

is generated by the images of well-known idempotents of $\hat{\mathcal{H}}_{n-1}^{(0,k+1)}$ under the isomorphism deduced from φ (see [?]).

Indexation of a basis

There exists a basis for $H_{(k,1,\dots,1),n}$ indexed by the set of elements of the form $b_1 \dots b_{n-1}$, $b_i \in \{1, \bar{1}, \dots, n-1, \bar{n-1}\}$ with at most k barred integers and no increasing sequence of the latter. In particular, the following identity holds (see [1],PZ):

$$\dim(H_{(k,1,\dots,1),n}) = \sum_{i=0}^{\min(k,n-1)} \binom{n-1}{i}^2 (n-1-i)!.$$

References

- [1] Y. Demesmay, *Fused permutations algebras and degenerate affine Hecke algebras*, 2025.
- [2] J. Hu, A. Mathas, *Seminormal forms and cyclotomic quiver Hecke algebras of type A*, Math. Ann. 364 (2016), no. 3-4, 1189–1254.
- [3] L. Poulain d'Andecy and M. Zaimi, *Fused Hecke algebra and one-boundary algebras*, Pacific J. Math. 328 (2024), no. 1, 77–118.